AXISYMMETRIC LOADING OF A SPACE WITH A SPHERICAL CUT

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The first fundamental problem of elasticity theory for a spherical cut in an elastic space is solved in [1] in a general formulation. Practical use of the solution in [1] is rather difficult because of the indeterminacy of one of the constants in the expression for the stresses and displacements; the available conditions therefore turn out to be inadequate for its determination.

An analogous problem is solved in the present paper by a method different from that in [1]. The investigation is based on the integral relations obtained by applying the transformation of [2-4] to the axisymmetric elasticity theory problem for a sphere. The expressions for finding the functions which solve the first fundamental problem of elasticity theory also contain an indeterminate constant. The value of this constant is found by comparison with the solution obtained in [1].

As an illustration, a spherical cut under symmetric and antisymmetric uniform loads applied to its edges is examined. Closed formulas are obtained to determine the normal and tangential stresses outside the cut on a sphere whose radius equals the radius of the cut.

1. The general solution of an axisymmetric problem of elasticity theory in spherical coordinates can be represented in terms of two analytic functions $F_1(\zeta)$ and $F_2(\zeta)$ of the complex variable $\zeta = \rho e^{i\theta}$

$$2G\frac{U_R}{R} = \frac{1}{\pi i}\int_{\bar t}^{t}\left[\left(\zeta\frac{d}{d\zeta} - 2 + 4\nu\right)F_1 + \frac{1}{R^4}\zeta F_2'\right]\frac{d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar t)}}$$

$$2G\frac{U_\theta}{R} = \frac{1}{\pi i r}\int_{\bar t}^{t}\left[\left(\zeta\frac{d}{d\zeta} + 5 - 4\nu\right)F_1 + \frac{1}{R^2}(\zeta F_2' + F_2)\right]\frac{(\zeta - z)\,d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar t)}} \qquad (1.1)$$

$$\sigma_R = \frac{1}{\pi i}\int_{\bar t}^{t}\left[\left(\zeta^2\frac{d^2}{d\zeta^2} - 2 - 2\nu\right)F_1 + \frac{1}{R^2}\zeta^2 F_2''\right]\frac{d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar t)}}$$

$$\tau_{R\theta} = \frac{1}{\pi i r}\int_{\bar t}^{t}\left[\left(\zeta^2\frac{d^2}{d\zeta^2} + 3\zeta\frac{d}{d\zeta} + 1 - 2\nu\right)F_1 + \right.$$
$$\left. + \frac{1}{R^2}\left(\zeta^2\frac{d}{d\zeta^2} + \zeta\frac{d}{d\zeta} - 1\right)F_2\right]\frac{(\zeta - z)\,d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar t)}} \qquad (1.2)$$

Assuming that

$$F_1 = a_n(n+1)\zeta^n,\ F_2 = b_n\zeta^n \quad \text{or} \quad F_1 = c_n n\zeta^{-n-1},\ F_2 = d_n\zeta^{-n-1}$$

we obtain the familiar formulas for the solution of the interior and exterior problems, respectively, on the axisymmetric straining of an elastic sphere [5].

The expansions of the functions F_1 and F_2 at infinity have the form

$$F_1 = c_{-2}\zeta^{-2} + c_{-3}\zeta^{-3} + \ldots, \quad F_2 = d_{-1}\zeta^{-1} + d_{-2}\zeta^{-2} + \ldots \qquad (1.3)$$

The constant can be determined in terms of the principal vector of the forces Z^* applied to a domain isolated within the space containing the loading surface

$$Z^* = 8\pi (1 - \nu) c_{-2} \tag{1.4}$$

Let us consider the elasticity theory problem for a space with a spherical cut of radius $R = 1$ and angle α (Fig. 1).

The axisymmetric normal $p^+(\theta)$, $p^-(\theta)$ and tangential $q^+(\theta)$, $q^-(\theta)$ loads are applied to the edges of the cut. For $R = 1$ we have

$$p^\pm = \pm \frac{2}{\pi} \int_0^\theta \operatorname{Re}[A^\pm(\sigma) e^{1/2 i\vartheta}] \frac{d\vartheta}{\sqrt{2(\cos\vartheta - \cos\theta)}}$$

$$q^\mp \sin\theta = \mp \frac{2}{\pi} \int_0^\theta \operatorname{Re}[B^\pm(\sigma) e^{1/2 i\vartheta}] \sqrt{2(\cos\vartheta - \cos\theta)}\, d\vartheta \tag{1.5}$$

at the edges of the cut ($\sigma = e^{i\vartheta}$).

Here $A(\zeta)$, $B(\zeta)$ are analytic functions of the variable ζ defined by the expressions

$$A(\zeta) = \zeta^2 F_1'' - 2(1+\nu) F_1 + \zeta^2 F_2''$$
$$B(\zeta) = \zeta [\zeta^2 F_1'' + 3\zeta F_1' - (1 - 2\nu) F_1 + \zeta^2 F_2'' + \zeta F_2' - F_2]' \tag{1.6}$$

By the condition of boundedness of the displacements on the cut boundary $R = 1$, $\theta = \alpha$, the following conditions must be satisfied

$$A(\zeta) = K_1 |\zeta - c|^{-\delta}$$
$$B(\zeta) = K_2 |\zeta - c|^{-\delta-1} \quad \delta < 3/2 \tag{1.7}$$

for $A(\zeta)$ and $B(\zeta)$.

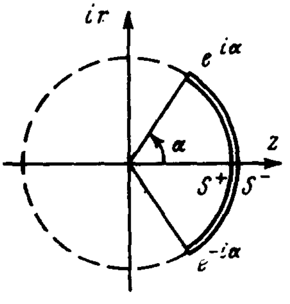

Fig. 1

Here K_1, K_2 are some positive constants, and c is either of the ends of the cut.

The expansions of $A(\zeta)$ and $B(\zeta)$ at zero and infinity are determined by the behavior of the functions F_1 and F_2

$$\lim_{\zeta\to\infty} A(\zeta) = O(\zeta^{-1}), \qquad \lim_{\zeta\to\infty} B(\zeta) = O(\zeta^{-2})$$
$$\lim_{\zeta\to 0} A(\zeta) = O(\zeta), \qquad \lim_{\zeta\to 0} B(\zeta) = \text{const} + O(\zeta) \tag{1.8}$$

For known $A(\zeta)$ and $B(\zeta)$ the function $F_1(\zeta)$ can be determined from the differential equation

$$B - \zeta A' - A = 2\zeta^2 F_1'' + 4(1+\nu)\zeta F_1' + 2(1+\nu) F_1 \tag{1.9}$$

Taking account of (1.4), we find from (1.9)

$$\lim_{\zeta\to\infty} (B - \zeta A' - A) = \frac{4Z^*}{4\pi\zeta^2} + O(\zeta^{-3}) \tag{1.10}$$

We now resolve the problem under consideration into its symmetric ($p^+ = p^- = p$, $q^+ = q^- = q$) and antisymmetric ($p^+ = -p^- = p$, $q^+ = -q^- = q$). components.

2. For symmetric loading of the cut conditions (1.5) become

$$p = \pm \frac{2}{\pi} \int_0^\theta \frac{\operatorname{Re} A^\pm(\sigma) e^{1/2 i\vartheta} d\vartheta}{\sqrt{2(\cos\vartheta - \cos\theta)}} \qquad (\theta \leqslant \alpha) \tag{2.1}$$

$$-q \sin\theta = \pm \frac{2}{\pi} \int_0^\theta \operatorname{Re}[B^\pm(\sigma) e^{1/2 i\vartheta}] \sqrt{2(\cos\vartheta - \cos\theta)}\, d\vartheta \tag{2.2}$$

Solving the integral equations (2.1), (2.2), we obtain

$$\operatorname{Re} A^+ e^{1/2 i\vartheta} = g_1(\vartheta), \qquad \operatorname{Re} A^- e^{1/2 i\vartheta} = -g_1(\vartheta) \tag{2.3}$$

$$\operatorname{Re} B^+ e^{1/2 i\vartheta} = g_2(\vartheta), \qquad \operatorname{Re} B^- e^{1/2 i\vartheta} = -g_2(\vartheta) \tag{2.4}$$

Here

$$g_1(\vartheta) = \frac{d}{d\vartheta}\int_0^{\vartheta} \frac{p(\theta)\sin\theta\, d\theta}{\sqrt{2(\cos\theta - \cos\vartheta)}}, \qquad g_1(\vartheta) = g_1(-\vartheta) \tag{2.5}$$

$$g_2(\vartheta) = \frac{d}{d\vartheta}\sin^{-1}\vartheta\,\frac{d}{d\vartheta}\int_0^{\vartheta} \frac{q(\theta)\sin^2\theta\, d\theta}{\sqrt{2(\cos\theta - \cos\vartheta)}}, \qquad g_2(\vartheta) = g_2(-\vartheta) \tag{2.6}$$

Problems (2.3), (2.4) can be reduced to the corresponding problems of linear conjugation. In particular, to solve problem (2.3), we introduce into S^- the function Ω related to $A(\zeta)$ by the expression $\Omega(\zeta) = \bar{A}(\zeta^{-1})$.

Relations (2.1) now become

$$\sigma A^+ + \Omega^- = g_1 e^{1/2 i\vartheta}, \qquad \sigma A^- + \Omega^+ = -g_1 e^{1/2 i\vartheta} \tag{2.7}$$

Problem (2.7) can be solved by known methods [6]. Recalling (1.7), (1.8), we obtain

$$A(\zeta) = \frac{1}{2\pi i}\int_{\bar{a}}^{a} g_1 e^{1/2 i\vartheta}\frac{d\sigma}{\sigma - \zeta} + \frac{a_0(\zeta - 1)}{X^2(\zeta)} \tag{2.8}$$

$$a = e^{i\alpha}, \qquad X(\zeta) = \sqrt{(\zeta - a)(\zeta - \bar{a})}, \qquad \lim_{\zeta\to 0} X(\xi) = 1$$

Similarly for (2.4) we obtain

$$B(\zeta) = \frac{1}{2\pi i}\int_{\bar{a}}^{a} g_2 e^{-1/2 i\vartheta}\frac{d\sigma}{\sigma - \zeta} + \frac{\zeta - 1}{2\pi i X^2(\zeta)}\int_{\bar{a}}^{a} g_2 e^{-1/2 i\vartheta}\, d\sigma + b_0\frac{\zeta(\zeta - 1)}{X^4(\zeta)} \tag{2.9}$$

We can find the relationship between a_0 and b_0 from (1.9) and (1.10)

$$b_0 + a_0(2\cos\alpha - 1) = \frac{1}{2\pi i}\int_{\bar{a}}^{a}[g_1\sigma + g_2(\sigma + 1 - 2\cos\alpha)]\,\sigma^{-1/2}\, d\sigma \tag{2.10}$$

In order to find the second condition for determining a_0 and b_0, we can proceed as follows. Following [1], we introduce the analytic function $A_1(\zeta)$, where

$$p_z = \frac{2}{\pi}\int_0^{\theta} \frac{\operatorname{Re} A_1(\sigma)\, e^{1/2 i\vartheta}\, d\vartheta}{\sqrt{2(\cos\vartheta - \cos\theta)}} \qquad (\theta < \alpha) \tag{2.11}$$

$$p_z = p\cos\theta - q\sin\theta \tag{2.12}$$

The method of determining $A_1(\zeta)$ is described in [1]. The appropriate calculations yield

$$A_1(\zeta) = \frac{1}{2\pi i}\int_{\bar{a}}^{a} g_3 e^{-1/2 i\vartheta}\frac{d\sigma}{\sigma - \zeta} + \frac{\zeta - 1}{2\pi i X^2(\zeta)}\int_{\bar{a}}^{a} g_3 e^{-1/2 i\vartheta}\, d\vartheta \tag{2.13}$$

$$g_3(\vartheta) = g_3(-\vartheta) = \frac{d}{d\vartheta}\int_0^{\vartheta} \frac{p_z(\theta)\sin\theta\, d\theta}{\sqrt{2(\cos\theta - \cos\vartheta)}} \tag{2.14}$$

By virtue of (2.12), the following condition must be satisfied on the sphere $R = 1$:

$$\operatorname{Re}\int_0^{\theta} \frac{A(\sigma)\cos\theta + 2(\cos\vartheta - \cos\theta)B(\sigma) - A_1(\sigma)}{\sqrt{2(\cos\vartheta - \cos\theta)}}\, e^{1/2 i\vartheta}\, d\vartheta = 0 \tag{2.15}$$

Condition (2.15) together with (2.10) enables us to determine the constants a_0 and b_0.
To illustrate, let us consider a spherical cut with the angle α under the uniform pressure p. Applying (2.3)–(2.6), we find that

$$\operatorname{Re} A^+ e^{1/2 i\theta} = p\cos\theta/2, \qquad \operatorname{Re} A^- e^{1/2 i\theta} = -p\cos\theta/2$$

$$\operatorname{Re} B^+ e^{1/2 i\theta} = 0, \qquad \operatorname{Re} B^- e^{1/2 i\theta} = 0$$

for $R = 1$.
We determine $A(\zeta)$ and $B(\zeta)$ from formulas (2.8),

$$A(\zeta) = \frac{p}{2\pi i}\left[\left(1+\frac{1}{\zeta}\right)\ln\frac{\zeta-a}{\zeta-\bar{a}} - \frac{2i\alpha}{\zeta}\right] + \frac{a_0(\zeta-1)}{X^2(\zeta)}$$

$$B(\zeta) = \frac{b_0\zeta(\zeta-1)}{X^4(\zeta)}, \quad b_0 + a_0(2\cos\alpha - 1) = \frac{p}{\pi}\sin\alpha(1+\cos\alpha)$$

We now have

$$I_1 = \frac{1}{\pi i}\int_{\bar{t}}^{t}\frac{A(\zeta)\,d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar{t})}}, \qquad I_2 = \frac{1}{\pi i}\int_{\bar{t}}^{t} B(\zeta)\sqrt{(\zeta-t)(\zeta-\bar{t})}\,d\zeta \tag{2.16}$$

Integrals (2.16) can be computed by the Muskhelishvili method

$$I_1 = \frac{p}{\pi}\operatorname{Im}\Big\{\ln\left[X(a,t) + a - R\cos\theta\right] -$$

$$-\frac{1}{R}\ln\bar{a}\left[X(a,t) + R - a\cos\theta\right] + a_0\operatorname{Im}\frac{1-a}{\sin\alpha X(a,t)}$$

$$I_2 = b_0\left[\operatorname{Im}\frac{\cos\alpha - 1}{2\sin^3\alpha}X(a,t) - \operatorname{Re}\frac{(a-1)(a-z)}{2\sin^2\alpha X(a,t)}\right]$$

Here

$$X(a,t) = \sqrt{(a-t)(a-\bar{t})} = \sqrt{R^2 - 2az + a^2}, \quad \lim_{R\to 0} X(a,t) = a$$

Similarly, (2.13) enables us to find $A_1(\zeta)$, and

$$I_3 = (\pi i)^{-1}\int_{\bar{t}}^{t} A_1\left[(\zeta-t)(\zeta-\bar{t})\right]^{-1/2} d\zeta =$$

$$= \frac{p}{\pi}\operatorname{Im}\Big\{\left(1-\frac{1}{aR^2}\right)X(a,t) + z\ln\left[X(a,t) + a - z\right] -$$

$$-\frac{\cos\theta}{R^2}\ln\left[\bar{a}X(a,t) + \bar{a}R - \cos\theta\right] + (1+\cos\alpha)\frac{1-a}{X(a,t)}\Big\}$$

Utilizing (2.15), after some simplifications, we obtain

$$-\frac{2p}{\pi}\sin^3\alpha\left(\sin^2\frac{\alpha}{2} + \cos\theta\right) + b_0\sin^2\frac{\alpha}{2}(1+\cos\theta) + a_0\sin^2\alpha\cos\theta = 0 \tag{2.17}$$

Equation (2.17) is satisfied only if

$$a_0 = \frac{2p}{\pi}\cos^2\frac{\alpha}{2}\sin\alpha, \qquad b_0 = \frac{2p}{\pi}\sin^3\alpha$$

It is easy to see that for given a_0 and b_0 condition (2.15) is satisfied identically. The stresses over the portion $R = 1$, $\theta > \alpha$ of the sphere can be determined in closed form. Appropriate calculations yield

$$\sigma_R = \frac{2p}{\pi}\left[\operatorname{arc\,tg}\frac{\sqrt{2}\sin\alpha/2}{\sqrt{\cos\alpha-\cos\theta}} - \frac{\sin\alpha\cos\alpha/2}{\sqrt{2(\cos\alpha-\cos\theta)}}\right]$$

$$\tau_{R\theta} = -\frac{4p\sin^3\alpha/2}{\pi\sin\theta}\,\frac{1+\cos\theta}{\sqrt{2(\cos\alpha-\cos\theta)}} \tag{2.18}$$

In the limiting case of a plane cut, (2.18) become the known formulas for the stress distribution in a plane coincident with the plane $z=0$ of the cut [7].

For antisymmetric loading, calculations analogous to those above yield: (3.1)

$$A=\frac{1}{2\pi iX(\zeta)}\int_{\bar a}^{a} g_1 e^{-1/2\, i\vartheta} X^+(\sigma)\left(1+\frac{\zeta}{\sigma}\right)\frac{d\sigma}{\sigma-\zeta}+\frac{a_0(\zeta-1)}{2\pi iX^2(\zeta)}\int_{\bar a}^{a} g_1 e^{-3/2\, i\vartheta} X^+(\sigma)d\sigma$$

$$B=\frac{1}{2\pi iX(\zeta)}\int_{\bar a}^{a} g_2 e^{-3/2\, i\vartheta} X^+(\sigma)\left(1+\frac{\zeta}{\sigma}\right)\frac{d\sigma}{\sigma-\zeta}+\frac{\zeta^2-1}{2\pi iX^3(\zeta)}\int_{\bar a}^{a} g_2 e^{-3/2\, i\vartheta} X^+(\sigma)d\sigma+\frac{b_0\zeta(\zeta-1)}{X^4(\zeta)}$$

Applying the results of [1], we obtain

$$A_1=\frac{1}{2\pi iX(\zeta)}\int_{\bar a}^{a}\frac{g_3}{\sqrt{\sigma}} X^+(\sigma)\left(1+\frac{\zeta}{\sigma}\right)\frac{d\sigma}{\sigma-\zeta}+\frac{\zeta^2-1}{4X^2(\zeta)}\int_{\bar a}^{a}\frac{g_3}{\sigma^{3/2}} X+(\sigma)\,d\sigma\Big] \qquad (3.2)$$

Here g_1, g_2 and g_3 are determined by (2.5), (2.6), (2.14), respectively.

The constants a_0 and b_0 can be determined from a joint analysis of (2.10) and condition (2.15).

To illustrate, let us consider the loading of a cut by an antisymmetric uniform pressure p. In this case

$$\mathrm{Re}A^+e^{1/2\, i\vartheta}=\mathrm{Re}A^-e^{1/2\, i\vartheta}=p\cos{}^1/_2\vartheta, \quad \mathrm{Re}B^+e^{1/2\, i\vartheta}=\mathrm{Re}\,B^-e^{1/2\, i\vartheta}=0$$

Formulas (3.1) and (3.2) yield

$$A=\frac{p}{2}\left[1+\frac{1}{\zeta}+\frac{\zeta-\zeta^{-1}}{X(\zeta)}\right]+\frac{a_0(\zeta-1)}{X^2(\zeta)}, \qquad B=\frac{b_0\zeta(\zeta-1)}{X^4(\zeta)}$$

$$A_1(\zeta)=\frac{p}{2}\left[\zeta+\frac{1}{\zeta^2}+\frac{\zeta+\zeta^{-2}+\cos\alpha(\zeta-\zeta^{-1})}{X(\zeta)}+\frac{\sin^2\alpha}{2X^2(\zeta)}\right]$$

From (1.1) we have

$$b_0+a_0(2\cos\alpha-1)=0.75\quad p\sin^2\alpha \qquad (3.3)$$

After some simple calculations, (2.15) gives us the equation

$$^1/_4\cos^4\alpha=a_0\cos\theta\sin^2\alpha+b_0\sin^2\alpha\,/\,2\;\;(1+\cos\theta)$$

The constants a_0 and b_0 are

$$b_0=p\cos^2\alpha\,/\,2,\;\; 4a_0=-p\sin^2\alpha$$

It is easy to see that condition (3.3) is satisfied. The stresses σ_R and $\tau_{R\theta}$ are determined for $R=1$, $\theta>\alpha$ by the dependences

$$\sigma_R=-\frac{p\sin^2\alpha}{4\sqrt{2(\cos\alpha-\cos\theta)}}, \quad \tau_{R\theta}=\frac{p\sin^2\alpha\cos^2\alpha/2}{\sin\theta\sqrt{2(\cos\alpha-\cos\theta)}}\;(1+\cos\theta)$$

Passage to the limiting case of a plane slit is not valid in this case, since in the limit outside the cut the stress σ_z in the plane of the slit is equal to zero, and the integral

$$\int_{r_0}^{\infty}\sigma_z\,|_{z=0}\,dr$$

taken in this plane has a constant value. Hence, the antisymmetric loading of a plane slit must be considered separately [8, 9].

BIBLIOGRAPHY

1. Boiko, L. T., Ziuzin, V. A. and Mossakovskii, V. I., A spherical cut in an elastic space. Dokl. Akad. Nauk SSSR, Vol. 181, №6, 1968.
2. Aleksandrov, A. Ia. and Vol'pert, V. S., On application of a method of solving axisymmetric elasticity theory problems to the problem of a sphere

and of a space with a spherical cavity. Izv. Akad. Nauk SSSR, OTN, Mekh. i Mashinostr. №6, 1961.
3. Polozhii, G. N., Generalization of the Theory of Analytic Functions of a Complex Variable. P-Analytic and (p, q)-Analytic Functions and Some of Their Applications. Kiev, Kiev University Press, 1965.
4. Aleksandrov, A. Ia. and Solov'ev, Iu. I., One form of solution of three-dimensional axisymmetric problems of elasticity theory by means of functions of a complex variable and the solution of these problems for the sphere. PMM Vol. 26, №1, 1962.
5. Lur'e, A. I., Spatial Problems of Elasticity Theory. Moscow, Gostekhizdat, 1955.
6. Muskhelishvili, N. I., Singular Integral Equations. Boundary Value Problems of the Theory of Functions and Some of Their Applications to Mathematical Physics, 2nd Edition. Fizmatgiz, 1962.
7. Sneddon, I., Fourier Transforms, (Russian translation). Moscow, IIL, 1955.
8. Mossakovskii, V. I., The first fundamental problem of the theory of elasticity for a space with a plane circular cavity. PMM Vol. 19, №4, 1955.
9. Ufliand, Ia. S., Integral Transforms in Elasticity Theory Problems. Moscow-Leningrad, Akad. Nauk SSSR Press, Leningrad Section, 1963.

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ON THE PROBLEM OF VIBRATIONS OF A SLIGHTLY CAMBERED PLATE

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The influence of slight camber of the middle line of a transverse section on the natural vibrations frequency and mode of an infinitely long plate, clamped at the endfaces, which is vibrating under plane strain conditions, is examined on the basis of perturbation theory [1−4] in the special case when some frequency of vibration of the uncambered plate is double. The initial system is degenerate [2], a "small imperfection can cause a large effect" for it ([1], Vol. 1, Sect. 149). The problem under consideration is a particular case of the problem of the influence of a small change in shape on the vibrations of a shell having multiple natural frequencies.

A supplement to an assertion of the author of [5] on the separation of natural shell vibrations into quasi-transverse and quasi-tangential is also contained herein.

1. To determine the mode and frequencies in the case under consideration, we have from the general equations of shell vibrations [5]

$$(A^{(0)} + \psi A^{(1)} + \psi^2 A^{(2)})\,(v,\ w) = \lambda\,(v,\ w) \qquad (1.1)$$
$$A^{(\nu)} = \| a_{ij}^{(\nu)} \| \qquad \nu = 0, 1, 2 \qquad i, j = 1, 2$$

Let us present expressions for the nonzero elements of the matrix operators $A^{(\nu)}$

$$a_{11}^{(0)} = -\frac{d^2}{ds^2}, \qquad a_{22}^{(0)} = \frac{h_*^2}{3}\frac{d^4}{ds^4}, \qquad a_{12}^{(1)} = \frac{d\,(\varkappa\,\cdot)}{ds} - \frac{h_*^2}{3}\varkappa\frac{d^3}{ds^3}$$